

Subtraction games with FES sets of size 3

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Abstract

This paper extends the work done by Angela Siegel [1] on subtraction games in which the subtraction set is $\mathbb{N} \setminus X$ for some finite set X . Siegel proves that for any finite set X , the \mathcal{G} -sequence is ultimately arithmetic periodic, and that if $|X| = 1$ or 2 , then it is purely arithmetic periodic. This note proves that if $|X| = 3$ then the \mathcal{G} -sequence is purely arithmetic periodic. It is known that for $|X| \geq 4$ the sequence is not always purely arithmetic periodic.

1 Introduction

The *subtraction game* for a set $S \subset \mathbb{N}$ is played on a heap of counters. A move is to choose a number from S and remove that many counters from the heap. The set S is called the *subtraction set*. We concern ourselves here with *normal play*, that is, the last player able to make a move wins.

An *all-but subtraction game* is a subtraction game where the subtraction set consists of all but finitely many of the natural numbers. Formally, $S = \mathbb{N} \setminus X$ for some finite set $X \subset \mathbb{N}$. This X is called the *finite excluded subtraction set* or FES set.

As with many games, all-but subtraction games are not interesting when played in isolation; they are more interesting as part of a disjunctive sum. We are therefore interested in the numbers of the pile sizes so that we can use Sprague-Grundy Theory [2] to analyze the positions.

We use $\mathcal{G}(n)$ to mean the number of a pile of size n , that is, $\mathcal{G}(n) = \text{mex}\{\mathcal{G}(n-s) \mid s \in S, s \leq n\}$. We will also call this function the *nim sequence* of the game.

A nim sequence is *periodic* (or *ultimately periodic*) if there exist $n_0, p \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathcal{G}(n+p) = \mathcal{G}(n)$. If $n_0 = 0$ we say the nim sequence is *purely periodic*. It is known that if the subtraction set S is finite, then the nim sequence is periodic.

A nim sequence is *arithmetic periodic* (or *ultimately arithmetic periodic*) if there exist $n_0, p, s \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\mathcal{G}(n+p) = \mathcal{G}(n) + s$. We call n_0 the *preperiod*, p the *period*, and s the *saltus*. If $n_0 = 0$ we say the nim sequence is *purely arithmetic periodic*. Siegel proved [1] that the nim sequence of any all-but subtraction game is arithmetic periodic.

Siegel also showed that in an all-but subtraction game, if the size of the FES set is 1 or 2, then the nim sequence is purely arithmetic periodic. We show in this paper that the same is true if the FES set has size 3.

2 Background

In calculating the nim sequence, it is natural to repeatedly apply the definition of the \mathcal{G} function. However, for the proofs in this paper, we require a different algorithm which we will call the FES algorithm.

Algorithm 1. (FES Algorithm)

For $k = 0, 1, 2, \dots$ find all m such that $\mathcal{G}(m) = k$ as follows:

Let $n = \min\{i : \mathcal{G}(i) \text{ is unknown}\}$. $\mathcal{G}(n) = k$.

For $x \in X$ considered in increasing order,

If $\mathcal{G}(n+x)$ is unknown and for all $m < n+x$ with $\mathcal{G}(m) = k$ we have $(n+x) - m \in X$ then $\mathcal{G}(n+x) = k$.

Theorem 2. Algorithm 1 correctly calculates the nim sequence.

Proof. The proof is by induction. The induction hypothesis is that after k iterations the algorithm has corrected labeled all the positions that have a number in the set $\{0, 1, \dots, k-1\}$. We need to prove that the next iteration correctly computes the placement of the k 's. We know that position n must have a k in it, because its value (by induction) is $> k-1$, but it cannot be $> k$ because there are no k 's before it. The positions where it places the remaining k 's are precisely those where you cannot reach a k by making a move. Thus by the same argument they must have a value of k . \square

As an example, here are the first few steps of the algorithm carried out for the FES set $X = \{2, 3, 6, 8\}$

$\mathcal{G}(n)$	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
$\mathcal{G}(n)$	0	\sqsubset	0	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	0	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
$\mathcal{G}(n)$	0	1	0	1	\sqsubset	\sqsubset	\sqsubset	\sqsubset	0	1	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
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$\mathcal{G}(n)$	0	1	0	1	2	\sqsubset	2	\sqsubset	0	1	\sqsubset	\sqsubset	2	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
$\mathcal{G}(n)$	0	1	0	1	2	3	2	3	0	1	\sqsubset	\sqsubset	2	3	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
$\mathcal{G}(n)$	0	1	0	1	2	3	2	3	0	1	4	\sqsubset	2	3	\sqsubset	\sqsubset	4	\sqsubset	4	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
$\mathcal{G}(n)$	0	1	0	1	2	3	2	3	0	1	4	5	2	3	5	\sqsubset	4	5	4	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
<hr/>																						
	\vdots																					

Note that after the first $k-1$ steps of this algorithm have been carried out, if we want to find which piles have number k , we don't need to know which piles have which number, only which piles have numbers less than k . For this reason, after each step, we want to just think about n -values as either having a number or not yet having a number. Define the function $\mathcal{H}_k(n) : \mathbb{N} \rightarrow \{*, \sqsubset\}$ by $\mathcal{H}_k(n) = *$ iff $\mathcal{G}(n) < k$ and $\mathcal{H}_k(n) = \sqsubset$ iff $\mathcal{G}(n) \geq k$. For example, compare \mathcal{H}_3 to our state of knowledge after the third iteration of the FES algorithm.

$\mathcal{H}_3(n)$	*	*	*	*	*	\sqsubset	*	\sqsubset	*	*	\sqsubset	\sqsubset	*	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset	\sqsubset
n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21

Note that the beginning “chunk” is all *s and the end “chunk” is all blanks. We are interested in the middle “chunk,” the short interval in which \mathcal{H}_k takes on values of both $*$ and \sqsubset .

Lemma 3. *For a fixed k , let $n = \min\{i : \mathcal{H}_k(i) = \sqsubset\}$. Then for $m \geq n + \max(X)$, we have $\mathcal{H}_k(m) = \sqsubset$.*

Proof. Let $m \geq n + \max(X)$, let $\ell < k$ be a number, and let p be the smallest pile such that $\mathcal{G}(p) = \ell$. Then $p < n$. Then since $m \geq n + \max(X) > p + \max(X)$, we can move from a pile of size m to a pile of size p . This tells us that $\mathcal{G}(m) \neq \ell$. Thus $\mathcal{G}(m) \geq k$ so $\mathcal{H}_k(m) = \sqsubset$. \square

Definition 4. *The boundary pattern of \mathcal{H}_k is the sequence*

$\mathcal{H}_k(n), \mathcal{H}_k(n+1), \mathcal{H}_k(n+2), \dots, \mathcal{H}_k(n + \max(X) - 1)$ where $n = \min\{i : \mathcal{H}_k(i) = \sqsubset\}$.

These boundary patterns characterize the \mathcal{H}_k 's in that knowing the boundary pattern of \mathcal{H}_{k-1} is sufficient information to find the boundary pattern of \mathcal{H}_k . Note that the sequence of boundary patterns is ultimately

$$\begin{array}{cccccccccccccccc}
\mathcal{G}(i) & \sqcup & \sqcup & \ell & \sqcup & \sqcup & \sqcup & k-1 & \sqcup & \sqcup & \sqcup & k-1 & \sqcup & \sqcup & \ell & \sqcup & \sqcup & \sqcup & k-1 & \sqcup & \sqcup \\
i & & & & & & & n & & & & n+a & & & n+b & & & m+a+b & & n+a+b & & \\
& & & m & & & & m+a & & & & & & & m+a+b & & & & & & &
\end{array}$$

However, since $\ell < k-1$, this means $\mathcal{H}_\ell(m+a) = \sqcup$. This is a contradiction, because lemma 6 implies that $\mathcal{G}(m+a) = \ell$.

Now we know that given \mathcal{H}_k we can find which values of m have $\mathcal{G}(m) = k-1$ and from this we can construct \mathcal{H}_{k-1} . This means that from a given boundary pattern, we can uniquely determine the boundary pattern that proceeds it, and so our proof is complete. \square

Lemma 8. *If $b > a$ and $b \neq 2a$ then $\mathcal{G}_{\{a,b,2a\}} = \mathcal{G}_{\{a,2a\}}$.*

Proof. We show by induction that the only boundary patterns that arise have the form

$\sqcup \sqcup \sqcup \dots \text{***} \sqcup \sqcup \sqcup \dots \text{***}$. That is, i copies of \sqcup followed by $a-i$ copies of $*$, then another i copies of \sqcup and another $a-i$ copies of $*$.

Our first boundary pattern is blank and so it has the form desired form with $i = a$. From such a boundary pattern we run the FES algorithm. Take $n = \min\{i : \mathcal{G}(i) \text{ is unknown}\}$. Then $\mathcal{G}(n) = k$. Then $\mathcal{G}(n+a) = k$. Then $n+b$ has $n+a$ as an option so $\mathcal{G}(n+b) \neq k$. Then $\mathcal{G}(n+2a) = k$. The new boundary pattern is of the same form, with i decreased by 1 (mod a).

Since the FES algorithm does never sets $\mathcal{G}(n+b) = k$, it runs the same as it would if $X = \{a, 2a\}$. \square

Lemma 9. *If $b \neq 2a$ and $X = \{a, b, 2b\}$ then $\mathcal{G}_X = \mathcal{G}_{\{a\}}$.*

Proof. The argument is similar to the previous one. We show by induction the only boundary patterns that arise have the form $\sqcup \sqcup \sqcup \dots \text{***}$. That is, i copies of \sqcup followed by $a-i$ copies of $*$.

Our first boundary pattern is blank and so it has the form desired form with $i = a$. From such a boundary patterns we run the FES algorithm. Take $n = \min\{i : \mathcal{G}(i) \text{ is unknown}\}$. Then $\mathcal{G}(n) = k$. Then $\mathcal{G}(n+a) = k$. Then $n+b$ and $n+2b$ have $n+a$ as an option so $\mathcal{G}(n+b) \neq k$ and $\mathcal{G}(n+2b) \neq k$. The new boundary pattern is of the same form, with i decreased by 1 (mod a).

Since the FES algorithm does never sets $\mathcal{G}(n+b) = k$ nor $\mathcal{G}(n+2b) = k$, it runs the same as it would if $X = \{a\}$. \square

Lemma 10. *Let $X = \{a, b, c\}$ where $c \neq a+b$, $c \neq 2a$ and $c \neq 2b$. Then $\mathcal{G}_{\{a,b,c\}} = \mathcal{G}_{\{a,b\}}$.*

Proof. Assume not, and let n be the first pile on which they differ. That is, $k = \mathcal{G}_{\{a,b,c\}}(n) \neq \mathcal{G}_{\{a,b\}}(n) = \ell$ and $\mathcal{G}_{\{a,b,c\}}(m) = \mathcal{G}_{\{a,b\}}(m)$ for all $m < n$. The mex sets for $\mathcal{G}_{\{a,b,c\}}(n)$ and $\mathcal{G}_{\{a,b\}}(n)$ only differ by one element, $\mathcal{G}(n-c)$. Since that one element is causing the first difference, we must have $\mathcal{G}(n-c) = k$.

$$\begin{array}{cccccccccccccccc}
\mathcal{G}_{\{a,b\}}(i) & \sqcup & \sqcup & k & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \ell & \sqcup & \sqcup \\
i & & & n-c & & & & n-c+a & & & & & n-c+b & & n & & \\
\mathcal{G}_X(i) & \sqcup & \sqcup & k & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & k & \sqcup & \sqcup \\
i & & & n-c & & & & n-c+a & & & & & n-c+b & & n & &
\end{array}$$

If $\mathcal{G}((n-c)+a) = k$ then since $\mathcal{G}_X(n) = k$, we must have that $n-c+a$ is not an option of n . This means $n-a = n-c+a$ or $n-b = n-c+a$. However, this implies $c = 2a$ or $c = a+b$.

If $\mathcal{G}((n-c)+a) \neq k$, then in $\{a, b\}$ the options of $n-c+b$ are a superset of the options of $n-c$, so $\mathcal{G}_{\{a,b\}}(n-c+b) \geq k$. For $n-c < m < n-c+b$, $n-c$ is an option of m , so $\mathcal{G}_{\{a,b\}}(m) \neq k$. Since $n-c$ is not an option of $n-c+b$ we have $\mathcal{G}_{\{a,b\}}(n-c+b) = k$.

$$\begin{array}{cccccccccccccccc}
\mathcal{G}_{\{a,b\}}(i) & \sqcup & \sqcup & k & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & k & \sqcup & \sqcup & \ell & \sqcup & \sqcup \\
i & & & n-c & & & & n-c+a & & & & & n-c+b & & n & & \\
\mathcal{G}_X(i) & \sqcup & \sqcup & k & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & \sqcup & k & \sqcup & \sqcup & k & \sqcup & \sqcup \\
i & & & n-c & & & & n-c+a & & & & & n-c+b & & n & &
\end{array}$$

However now we have that $n - c + b$ is not an option of n , so $n - a = n - c + b$ or $n - b = n - c + b$. This implies that $c = a + b$ or $c = 2b$. □

Theorem 11. *If $|X| = 3$ then \mathcal{G}_X is purely arithmetic periodic.*

Proof. The proof breaks down into three cases, which are taken care of in the previous three lemmas. □

4 Conjectures & Future Work

The proof that $\mathcal{G}_{\{a,b,a+b\}}$ is purely periodic gives no insight into the length of the period. However there seem to be some obvious patterns in the case where $b > 3a$. Lemma 6 implies that the period is three times the saltus, so we only need to consider the saltus. It is known that the period for $\{na, nb, n(a+b)\}$ is n times the period for $\{a, b, a+b\}$ so we only need to consider FES sets where a and b are relatively prime.

Conjecture 12. *Let a and b be such that $b > 3a$ and $\gcd(a, b) = 1$ and let p be the period of the nim sequence for the FES set $\{a, b, a+b\}$. If there exists an m that is a multiple of $2a$ with $b < m < a + b$ $p = 3am$. If no such m exists, then there is some other n with $b < n < a + b$ such that $p = 3an$.*

The following pages contain the data from which this conjecture was drawn.

FES set	saltus	FES set	saltus	FES set	saltus	FES set	saltus
{1, 3, 4}	1 * 4	{2, 7, 9}	2 * 8	{3, 10, 13}	3 * 12	{4, 13, 17}	4 * 16
{1, 4, 5}	1 * 4	{2, 9, 11}	2 * 10	{3, 11, 14}	3 * 12	{4, 15, 19}	4 * 16
{1, 5, 6}	1 * 6	{2, 11, 13}	2 * 12	{3, 13, 16}	3 * 14	{4, 17, 21}	4 * 18
{1, 6, 7}	1 * 6	{2, 13, 15}	2 * 14	{3, 14, 17}	3 * 16	{4, 19, 23}	4 * 22
{1, 7, 8}	1 * 8	{2, 15, 17}	2 * 16	{3, 16, 19}	3 * 18	{4, 21, 25}	4 * 24
{1, 8, 9}	1 * 8	{2, 17, 19}	2 * 18	{3, 17, 20}	3 * 18	{4, 23, 27}	4 * 24
{1, 9, 10}	1 * 10	{2, 19, 21}	2 * 20	{3, 19, 22}	3 * 20	{4, 25, 29}	4 * 26
{1, 10, 11}	1 * 10	{2, 21, 23}	2 * 22	{3, 20, 23}	3 * 22	{4, 27, 31}	4 * 30
{1, 11, 12}	1 * 12	{2, 23, 25}	2 * 24	{3, 22, 25}	3 * 24	{4, 29, 33}	4 * 32
{1, 12, 13}	1 * 12	{2, 25, 27}	2 * 26	{3, 23, 26}	3 * 24	{4, 31, 35}	4 * 32
{1, 13, 14}	1 * 14	{2, 27, 29}	2 * 28	{3, 25, 28}	3 * 26	{4, 33, 37}	4 * 34
{1, 14, 15}	1 * 14	{2, 29, 31}	2 * 30	{3, 26, 29}	3 * 28	{4, 35, 39}	4 * 38
{1, 15, 16}	1 * 16	{2, 31, 33}	2 * 32	{3, 28, 31}	3 * 30	{4, 37, 41}	4 * 40
{1, 16, 17}	1 * 16	{2, 33, 35}	2 * 34	{3, 29, 32}	3 * 30	{4, 39, 43}	4 * 40
{1, 17, 18}	1 * 18	{2, 35, 37}	2 * 36	{3, 31, 34}	3 * 32	{4, 41, 45}	4 * 42
{1, 18, 19}	1 * 18	{2, 37, 39}	2 * 38	{3, 32, 35}	3 * 34	{4, 43, 47}	4 * 46
{1, 19, 20}	1 * 20	{2, 39, 41}	2 * 40	{3, 34, 37}	3 * 36	{4, 45, 49}	4 * 48
{1, 20, 21}	1 * 20	{2, 41, 43}	2 * 42	{3, 35, 38}	3 * 36	{4, 47, 51}	4 * 48
{1, 21, 22}	1 * 22	{2, 43, 45}	2 * 44	{3, 37, 40}	3 * 38	{4, 49, 53}	4 * 50
{1, 22, 23}	1 * 22	{2, 45, 47}	2 * 46	{3, 38, 41}	3 * 40	{4, 51, 55}	4 * 54
{1, 23, 24}	1 * 24	{2, 47, 49}	2 * 48	{3, 40, 43}	3 * 42	{4, 53, 57}	4 * 56
{1, 24, 25}	1 * 24	{2, 49, 51}	2 * 50	{3, 41, 44}	3 * 42	{4, 55, 59}	4 * 56
{1, 25, 26}	1 * 26	{2, 51, 53}	2 * 52	{3, 43, 46}	3 * 44	{4, 57, 61}	4 * 58
{1, 26, 27}	1 * 26	{2, 53, 55}	2 * 54	{3, 44, 47}	3 * 46	{4, 59, 63}	4 * 62
{1, 27, 28}	1 * 28	{2, 55, 57}	2 * 56	{3, 46, 49}	3 * 48	{4, 61, 65}	4 * 64
{1, 28, 29}	1 * 28	{2, 57, 59}	2 * 58	{3, 47, 50}	3 * 48	{4, 63, 67}	4 * 64
{1, 29, 30}	1 * 30	{2, 59, 61}	2 * 60	{3, 49, 52}	3 * 50	{4, 65, 69}	4 * 66
{1, 30, 31}	1 * 30	{2, 61, 63}	2 * 62	{3, 50, 53}	3 * 52	{4, 67, 71}	4 * 70
{1, 31, 32}	1 * 32	{2, 63, 65}	2 * 64	{3, 52, 55}	3 * 54	{4, 69, 73}	4 * 72
{1, 32, 33}	1 * 32	{2, 65, 67}	2 * 66	{3, 53, 56}	3 * 54	{4, 71, 75}	4 * 72
{1, 33, 34}	1 * 34	{2, 67, 69}	2 * 68	{3, 55, 58}	3 * 56	{4, 73, 77}	4 * 74
{1, 34, 35}	1 * 34	{2, 69, 71}	2 * 70	{3, 56, 59}	3 * 58	{4, 75, 79}	4 * 78
{1, 35, 36}	1 * 36	{2, 71, 73}	2 * 72	{3, 58, 61}	3 * 60	{4, 77, 81}	4 * 80
{1, 36, 37}	1 * 36	{2, 73, 75}	2 * 74	{3, 59, 62}	3 * 60	{4, 79, 83}	4 * 80
{1, 37, 38}	1 * 38	{2, 75, 77}	2 * 76	{3, 61, 64}	3 * 62	{4, 81, 85}	4 * 82
{1, 38, 39}	1 * 38	{2, 77, 79}	2 * 78	{3, 62, 65}	3 * 64	{4, 83, 87}	4 * 86
{1, 39, 40}	1 * 40	{2, 79, 81}	2 * 80	{3, 64, 67}	3 * 66	{4, 85, 89}	4 * 88
{1, 40, 41}	1 * 40	{2, 81, 83}	2 * 82	{3, 65, 68}	3 * 66	{4, 87, 91}	4 * 88
{1, 41, 42}	1 * 42	{2, 83, 85}	2 * 84	{3, 67, 70}	3 * 68	{4, 89, 93}	4 * 90
{1, 42, 43}	1 * 42	{2, 85, 87}	2 * 86	{3, 68, 71}	3 * 70	{4, 91, 95}	4 * 94

FES set	saltus	FES set	saltus	FES set	saltus	FES set	saltus
{5, 16, 21}	5 * 20	{6, 19, 25}	6 * 24	{7, 22, 29}	7 * 28	{8, 25, 33}	8 * 32
{5, 17, 22}	5 * 20	{6, 23, 29}	6 * 24	{7, 23, 30}	7 * 28	{8, 27, 35}	8 * 32
{5, 18, 23}	5 * 20	{6, 25, 31}	6 * 26	{7, 24, 31}	7 * 28	{8, 29, 37}	8 * 32
{5, 19, 24}	5 * 20	{6, 29, 35}	6 * 34	{7, 25, 32}	7 * 28	{8, 31, 39}	8 * 32
{5, 21, 26}	5 * 22	{6, 31, 37}	6 * 36	{7, 26, 33}	7 * 28	{8, 33, 41}	8 * 34
{5, 22, 27}	5 * 24	{6, 35, 41}	6 * 36	{7, 27, 34}	7 * 28	{8, 35, 43}	8 * 38
{5, 23, 28}	5 * 26	{6, 37, 43}	6 * 38	{7, 29, 36}	7 * 30	{8, 37, 45}	8 * 42
{5, 24, 29}	5 * 28	{6, 41, 47}	6 * 46	{7, 30, 37}	7 * 32	{8, 39, 47}	8 * 46
{5, 26, 31}	5 * 30	{6, 43, 49}	6 * 48	{7, 31, 38}	7 * 34	{8, 41, 49}	8 * 48
{5, 27, 32}	5 * 30	{6, 47, 53}	6 * 48	{7, 32, 39}	7 * 36	{8, 43, 51}	8 * 48
{5, 28, 33}	5 * 30	{6, 49, 55}	6 * 50	{7, 33, 40}	7 * 38	{8, 45, 53}	8 * 48
{5, 29, 34}	5 * 30	{6, 53, 59}	6 * 58	{7, 34, 41}	7 * 40	{8, 47, 55}	8 * 48
{5, 31, 36}	5 * 32	{6, 55, 61}	6 * 60	{7, 36, 43}	7 * 42	{8, 49, 57}	8 * 50
{5, 32, 37}	5 * 34	{6, 59, 65}	6 * 60	{7, 37, 44}	7 * 42	{8, 51, 59}	8 * 54
{5, 33, 38}	5 * 36	{6, 61, 67}	6 * 62	{7, 38, 45}	7 * 42	{8, 53, 61}	8 * 58
{5, 34, 39}	5 * 38	{6, 65, 71}	6 * 70	{7, 39, 46}	7 * 42	{8, 55, 63}	8 * 62
{5, 36, 41}	5 * 40	{6, 67, 73}	6 * 72	{7, 40, 47}	7 * 42	{8, 57, 65}	8 * 64
{5, 37, 42}	5 * 40	{6, 71, 77}	6 * 72	{7, 41, 48}	7 * 42	{8, 59, 67}	8 * 64
{5, 38, 43}	5 * 40	{6, 73, 79}	6 * 74	{7, 43, 50}	7 * 44	{8, 61, 69}	8 * 64
{5, 39, 44}	5 * 40	{6, 77, 83}	6 * 82	{7, 44, 51}	7 * 46	{8, 63, 71}	8 * 64
{5, 41, 46}	5 * 42	{6, 79, 85}	6 * 84	{7, 45, 52}	7 * 48	{8, 65, 73}	8 * 66
{5, 42, 47}	5 * 44	{6, 83, 89}	6 * 84	{7, 46, 53}	7 * 50	{8, 67, 75}	8 * 70
{5, 43, 48}	5 * 46	{6, 85, 91}	6 * 86	{7, 47, 54}	7 * 52	{8, 69, 77}	8 * 74
{5, 44, 49}	5 * 48	{6, 89, 95}	6 * 94	{7, 48, 55}	7 * 54	{8, 71, 79}	8 * 78
{5, 46, 51}	5 * 50	{6, 91, 97}	6 * 96	{7, 50, 57}	7 * 56	{8, 73, 81}	8 * 80
{5, 47, 52}	5 * 50	{6, 95, 101}	6 * 96	{7, 51, 58}	7 * 56	{8, 75, 83}	8 * 80
{5, 48, 53}	5 * 50	{6, 97, 103}	6 * 98	{7, 52, 59}	7 * 56	{8, 77, 85}	8 * 80
{5, 49, 54}	5 * 50	{6, 101, 107}	6 * 106	{7, 53, 60}	7 * 56	{8, 79, 87}	8 * 80
{5, 51, 56}	5 * 52	{6, 103, 109}	6 * 108	{7, 54, 61}	7 * 56	{8, 81, 89}	8 * 82
{5, 52, 57}	5 * 54	{6, 107, 113}	6 * 108	{7, 55, 62}	7 * 56	{8, 83, 91}	8 * 86
{5, 53, 58}	5 * 56	{6, 109, 115}	6 * 110	{7, 57, 64}	7 * 58	{8, 85, 93}	8 * 90
{5, 54, 59}	5 * 58	{6, 113, 119}	6 * 118	{7, 58, 65}	7 * 60	{8, 87, 95}	8 * 94
{5, 56, 61}	5 * 60	{6, 115, 121}	6 * 120	{7, 59, 66}	7 * 62	{8, 89, 97}	8 * 96
{5, 57, 62}	5 * 60	{6, 119, 125}	6 * 120	{7, 60, 67}	7 * 64	{8, 91, 99}	8 * 96
{5, 58, 63}	5 * 60	{6, 121, 127}	6 * 122	{7, 61, 68}	7 * 66	{8, 93, 101}	8 * 96
{5, 59, 64}	5 * 60	{6, 125, 131}	6 * 130	{7, 62, 69}	7 * 68	{8, 95, 103}	8 * 96
{5, 61, 66}	5 * 62	{6, 127, 133}	6 * 132	{7, 64, 71}	7 * 70	{8, 97, 105}	8 * 98
{5, 62, 67}	5 * 64	{6, 131, 137}	6 * 132	{7, 65, 72}	7 * 70	{8, 99, 107}	8 * 102
{5, 63, 68}	5 * 66	{6, 133, 139}	6 * 134	{7, 66, 73}	7 * 70	{8, 101, 109}	8 * 106
{5, 64, 69}	5 * 68	{6, 137, 143}	6 * 142	{7, 67, 74}	7 * 70	{8, 103, 111}	8 * 110

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